

# Rank 72 high minimum norm lattices

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### Abstract

Given a polarization of an even unimodular lattice and integer  $k \geq 1$ , we define a family of unimodular lattices  $L(M, N, k)$ . Of special interest are certain  $L(M, N, 3)$  of rank 72. Their minimum norms lie in  $\{4, 6, 8\}$ . Norms 4 and 6 do occur. Consequently, 6 becomes the highest known minimum norm for rank 72 even unimodular lattices. We discuss how norm 8 might occur for such a  $L(M, N, 3)$ . We note a few  $L(M, N, k)$  in dimensions 96, 120 and 128 with moderately high minimum norms.

**Key words:** even unimodular lattice, extremal lattice, Leech lattice, fourvolution, polarization, high minimum norm.

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## 1 Introduction

Integral positive definite lattices with high norm for a given rank and discriminant have attracted a lot of attention, due to their connections with modular forms, number theory, combinatorics and group theory. Especially intriguing are those even unimodular lattices which are *extremal*, i.e. their minimum norms achieve the theoretical upper bound  $2(\lfloor \frac{n}{24} \rfloor + 1)$ , where  $n$  is the rank. The rank of an even unimodular lattices must be divisible by 8 (e.g., [16]). The rank of an even integral unimodular extremal lattice is bounded (see [1] or Chapter 7 of [4] and the references therein). Extremal lattices are known to exist in dimensions a multiple of 8 up through 80, except for dimension 72. An extremal rank 72 lattice would have minimum norm 8 [4, 1].

In this article, we construct a family of unimodular lattices  $L(M, N, k)$  (2.6) for an integer  $k$  and unimodular integral lattices  $M, N$  which form a polarization (2.3). Estimates on the minimum norm of  $L(M, N, k)$  give some new examples of lattices with moderately high minimum norms.

Of special interest are those  $L(M, N, 3)$  of dimension 72 where we input Niemeier lattices for  $M$  and  $N$ . Such a  $L(M, N, 3)$  have minimum norm 4, 6 or 8. Norms 4 and 6 occur. According to [14], our result is the first proof that there exists a rank 72 even unimodular lattice for which the minimum norm

is at least 6. We indicate a specific criterion to be checked for such  $L(M, N, 3)$  to have minimum norm 8. We conclude by noting certain  $L(M, N, k)$  with moderately high norms in dimensions 96, 120 and 128.

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## 2 Integral sublattices of $\Upsilon^3$

**Definition 2.1.** *<lattice>* In this article, *lattice* means a rational positive definite lattice. The term *even lattice* means an integral lattice in which all norms are integral. For a lattice  $L$ , we define  $\mu(L) := \min\{(x, x) \mid x \in L, x \neq 0\}$  and call it the *minimum norm of  $L$* . If  $L_1, L_2, \dots$  is a set of lattices, we define  $\mu(L_1, L_2, \dots)$  to be the minimum of  $\mu(L_1), \mu(L_2), \dots$ .

**Definition 2.2.** *<polarization>* Suppose that  $E$  is an integral unimodular lattice. A *polarization* is a pair of sublattices  $X, Y$  such that  $(X, X) \leq 2\mathbb{Z}$ ,  $(Y, Y) \leq 2\mathbb{Z}$ ,  $X + Y = E$  and  $X \cap Y = 2E$ . It follows that  $E$  is even. If  $E$  is a lattice and  $r > 0$  is a rational number such that  $\sqrt{r}E$  is an integral unimodular lattice, a *polarization* of  $E$  is a pair of sublattices  $X, Y$  so that  $\sqrt{r}X, \sqrt{r}Y$  is a polarization of  $\sqrt{r}E$ .

**Remark 2.3.** *<polarization2>* If  $Z$  is one of  $X, Y$  as in (2.2) and  $E$  is unimodular, then  $\frac{1}{\sqrt{2}}Z$  is integral and unimodular, but may not be even. If  $\frac{1}{\sqrt{2}}X$  and  $\frac{1}{\sqrt{2}}Y$  are both even lattices we call the polarization an *even polarization*. If  $E$  is not unimodular but  $\sqrt{r}E$  is, the polarization  $X, Y$  of  $E$  is called *even* if the polarization  $\sqrt{r}X, \sqrt{r}Y$  is even.

**Notation 2.4.** *<ups>* We let  $\Upsilon$  be a lattice so that  $U := \sqrt{2}\Upsilon$  is an even, integral unimodular lattice.

A polarization of  $\Upsilon$  is therefore a pair of integral sublattices  $M, N$  such that  $M + N = \Upsilon$  and  $M \cap N = 2\Upsilon$ .

For the time being,  $\text{rank}(\Upsilon) = \text{rank}(U)$  is an arbitrary multiple of 8. We know the complete list of possibilities for even, integral unimodular lattices only in dimensions 8, 16 and 24. The rank 24 lattices are called *Niemeier lattices* since they were first classified by Niemeier [15].

**Lemma 2.5.** <e8polar> *The  $E_8$ -lattice has an even polarization.*

**Proof.** This is a standard fact. It follows since the  $E_8$  lattice modulo 2 has a nonsingular form with maximal Witt index. One then quotes the characterization of  $E_8$  as the unique (up to isometry) rank 8 even unimodular lattice. Another proof uses the existence of a fourvolution (7.1) on  $E_8$  (one exists, for example, in a natural  $Weyl(D_8)$  subgroup; if one identifies  $E_8$  with  $BW_{2^3}$ , the natural group of isometries  $BW_{2^3}$  contains lower fourvolutions).  $\square$

**Notation 2.6.** <gen1> We use the notation of (2.4) and let  $M, N$  be a polarization of  $\Upsilon$ . Let  $k \geq 2$ . Define these sublattices of  $\Upsilon^k$ :

$$L_M := \{(x_1, \dots, x_k) \in M^k \mid x_1 + \dots + x_k \in M \cap N\},$$

$$L^N := \{(y, y, \dots, y) \mid y \in N\},$$

$$L(M, N, k) := L_M + L^N.$$

**Remark 2.7.** <gen1.5> Because  $L(M, N, 1) = N$  and  $L(M, N, 2) \cong U \perp U$ , the interesting case is  $k \geq 3$ . If  $k = 2q$  is even,  $L(M, N, k)$  contains  $L^M + L^N$ , a sublattice isometric to  $\sqrt{q}U$ .

**Proposition 2.8.** <gen2> (i) *The lattice  $L(M, N, k)$  is an integral lattice and the sublattice  $L_M$  is even.*

(ii) *If  $k$  is an even integer or  $N$  is an even lattice,  $L(M, N, k)$  is an even lattice. Otherwise,  $L(M, N, k)$  is odd.*

(iii)  *$L(M, N, k)$  is unimodular.*

**Proof.** (i) To prove integrality, one shows that  $L_M$  and  $L^N$  are integral lattices and that  $(L_M, L^N) \leq \mathbb{Z}$ . The latter follows since for  $(x_1, \dots, x_k) \in L_M$ ,  $\sum_i x_i \in N$ , an integral lattice. Finally, the evenness of  $L_M$  is obvious since it is integral and a set of generators is even (e.g., all vectors of the form  $(x, x, 0^{k-1})$ ,  $x \in M$  and  $(y, 0^{k-1})$ ,  $y \in 2\Upsilon$ ).

(ii) This is obvious from the definition of  $L^N$ .

(iii) To prove unimodularity, it suffices by (6.1) to show that  $|L : L_M|^2 = \det(L_M)$ . We have  $\det(L_M) = \det(M^k) |M^k : L_M|^2 = 1 \cdot 2^{\text{rank}(M)}$  and  $|L : L_M| = |L_M + L^N : L_M| = |L^N : L^N \cap L_M| = |L^N : L^N \cap M^k| |L^N \cap M^k : L^N \cap L_M| = 2^{\frac{1}{2}\text{rank}(M)} \cdot 1$ .  $\square$

**Theorem 2.9.**  $\langle \text{minlmn} \rangle$  We use the notation  $\mu(L_1, L_2, \dots)$  (2.1).

- (i)  $\mu(L_M) = 2\mu(M, U)$  and  $\mu(L^N) = k\mu(N)$ .
- (ii)  $\mu(L) \leq \min\{k\mu(N), 2\mu(M, U)\}$ .
- (iii)  $\mu(L) \geq \min\{\frac{k}{2}\mu(U), 2\mu(M, U)\}$ .

**Proof.** (i) To determine  $\mu(L_M)$ , consider the possibility that all entries of  $(x_1, \dots, x_k) \in L_M$  are in  $2\Upsilon$ .

(ii) This follows from (i) since  $L_M$  and  $L^N$  are sublattices of  $L$ .

(iii) If a vector is in  $L \setminus L_M$ , all of its coordinates are nonzero.  $\square$

**Notation 2.10.**  $\langle \text{leechdef} \rangle$  We let  $\Lambda$  be a *Leech lattice*, i.e., a Niemeier lattice without roots.

Uniqueness of a rootless Niemeier lattice was proved first in [3], then in different styles in [2] and [7].

We illustrate the use of (2.9) by constructing a Leech lattice. This argument comes from [17], [13]. An analogous construction of a Golay code was created earlier by Turyn [18]. The original existence proof of the Leech lattice [12] makes use of the Golay code (whereas (2.11) does not).

**Corollary 2.11.**  $\langle \text{leech} \rangle$  *Leech lattices exist.*

**Proof.** We take  $M \cong N \cong E_8$  (2.5). From (2.9),  $3 \leq \mu(L) \leq 4$ . Since  $L(M, N, 3)$  is even,  $\mu(L(M, N, 3)) = 4$ .  $\square$

**Notation 2.12.**  $\langle \text{leechnota} \rangle$  We use the standard notation  $\Lambda$  for a Leech lattice.

### 3 Minimum norms for rank 72 $L(M, N, 3)$

**Notation 3.1.**  $\langle \text{rank72nota} \rangle$  In this section,  $L(M, N, 3)$  is a rank 72 lattice for which  $M$  and  $N$  are Niemeier lattices.

The minimum norm of a Niemeier lattice is 2 unless it is the Leech lattice, for which the minimum norm is 4.

**Corollary 3.2.**  $\langle \text{mul72} \rangle$  (i)  $\mu(L(M, N, 3)) \geq 4$ .

(ii) If  $M \not\cong \Lambda$ , then  $\mu(L(M, N, 3)) = 4$ .

(iii) If  $U \cong M \cong \Lambda$ , then  $\mu(L(M, N, 3)) \geq 6$ .

(iv) If  $U \cong M \cong \Lambda$ , and  $N \not\cong \Lambda$ , then  $\mu(L(M, N, 3)) = 6$ .

We now prove that situations (ii) and (iv) of the Corollary actually occur. This means proof that suitable polarizations of  $\Upsilon$  exist.

**Proposition 3.3.** *There exist  $L(M, N, 3)$  with minimum norms 4 and 6.*

**Proof.** We take  $U \cong E_8^3$  and  $M, N \leq U, M \cong N \cong \sqrt{2}E_8^3$  such that  $M + N = U$  (for example, the orthogonal direct sum of three polarizations as in (2.11) will do). Then (ii) applies.

If  $U \cong \Lambda$ , take in  $\Upsilon$  any sublattice  $M \cong \Lambda$  (see (7.2), (7.3)) and any  $N \cong E_8^3$  (see [7] for existence). Then (iv) applies.  $\square$

**Corollary 3.4.** *If  $\mu(L(M, N, 3)) = 8$ ,  $M \cong N \cong \Lambda$ .*

The question remains whether there exists a polarization  $M, N$  so that  $\mu(L(M, N, 3)) = 8$ .

**Remark 3.5.** It would be useful to know more about embeddings of  $\sqrt{2}J$  into  $K$ , where  $J, K$  are Niemeier lattices. For the case  $K \cong \Lambda$ , see [5], Th. 4.1. Note also that embeddings of  $\sqrt{2}E_8^3$  in  $\Lambda$  were used extensively in [7].

## 4 Norm 6 vectors in rank 72 $L(M, N, 3)$

**Notation 4.1.** Let  $L := L(M, N, 3)$ , where  $M \cong N \cong \Lambda$  (by (7.3), there exists such a polarization).

From (3.2)(iii),  $\mu(L) \geq 6$ . We consider the possibility that  $L$  has vectors of norm 6 and derive some results about forms of norm 6 vectors.

We use parentheses both for inner products  $(x, y)$  and  $n$ -tuples  $(x_1, \dots, x_n)$ . We hope for no confusion when  $n = 2$ .

**Notation 4.2.** We call an ordered 4-tuple  $(w, x, y, z) \in N \times M \times M \times M$  *admissible* if  $x + y + z \in M \cap N$ . The elements of  $L$  are the  $(x + w, y + w, z + w)$ , for all admissible 4-tuples  $(w, x, y, z)$ . We call admissible 4-tuples  $(x, y, z, w)$  and  $(x', y', z', w')$  *equivalent* if  $(x + w, y + w, z + w) = (x' + w', y' + w', z' + w')$ . An *offender* is a 4-tuple  $(x, y, z, w)$  such that each of  $r_x := x + w, r_y := y + w, r_z := z + w$  has norm 2. Offenders are those admissible 4-tuples which give norm 6 vectors  $(x + w, y + w, z + w) \in L$  (since  $\mu(M) = 4$ ,  $w \notin M$  or else  $M$  would contain roots). The set  $r_x, r_y, r_z$  is called *a triple of offender roots*.

If there are no offenders,  $L$  has minimum norm 8. We therefore study hypothetical offenders.

The rational lattice  $\Upsilon = M + N$  is not integral (in fact,  $(\Upsilon, \Upsilon) = \frac{1}{2}\mathbb{Z}$ ). The next result asserts integrality of the sublattice of  $\Upsilon$  spanned by the components of an offender.

**Lemma 4.3.** *<offint> For an offender,  $(w, x, y, z)$ , we define  $K$  to be the  $\mathbb{Z}$ -span of  $w, x, y, z$ . Then*

- (i) *The image of  $K$  in  $(M + N)/M$  has order 2;*
- (ii)  *$K$  is an even integral lattice.*

**Proof.** (i) The image of  $K$  in  $(M + N)/M$  is spanned by the image of  $w$ , and  $w \notin M, 2w \in M$ .

(ii) Since  $x, y, z$  lie in an integral lattice  $M$  and  $w \in N$  is integral, it suffices to prove that each of  $(w, x), (w, y), (w, z)$  is integral. We have  $2 = (w + x, w + x) = (w, w) + 2(w, x) + (x, x)$ . Since  $M$  and  $N$  are even lattices,  $(w, w)$  and  $(x, x)$  are even integers. So  $(w, x)$  is integral. Similarly, we prove  $(w, y), (w, z)$  are integral.  $\square$

**Lemma 4.4.** *<shortmod> Let  $Q$  be a sublattice of  $\Lambda$ ,  $Q \cong \sqrt{2}\Lambda$ . The  $2^{12} - 1$  nontrivial cosets each contain exactly 48 norm 4 vectors, and such a set of 48 is an orthogonal frame: two members are proportional or orthogonal.*

**Proof.** This may be proved by a rescaling of the argument that in  $\Lambda$ , the norm 8 vectors which lie in the same coset of  $2\Lambda$  constitute an orthogonal frame of 48 vectors. See [3, 6].  $\square$

**Lemma 4.5.** *<wnorm4> Suppose that  $M$  has fourvolution type (7.2). If  $(w, x, y, z)$  is admissible and  $w \notin M$ , there exists an equivalent admissible quadruple  $(w', x', y', z')$  such that  $w'$  has norm 4.*

**Proof.** This follows from (4.4). There exists  $v \in \Upsilon$  so that  $w' := w - 2v \in N$  has norm 4 (recall that  $2\Upsilon = M \cap N$ ). Take  $x' := x + 2v, y' := y + 2v, z' := z + 2v$ . These three vectors lie in  $M$ .  $\square$

**Lemma 4.6.** *<orthogoffenderroots> A triple of offender roots is a pairwise orthogonal set.*

**Proof.** Suppose that two such roots are not orthogonal, say  $r = w + x$  and  $s = w + y$ . Define  $J := \text{span}\{r, s\}$ , an  $A_2$ -lattice (note that  $J$  is integral, by (4.3)(ii)). Since  $M \cap J$  is contained in  $M$ , it is rootless. However,  $M \cap J$  has index 2 in  $J$  gives a contradiction since every index 2 sublattice of  $J$  contains roots.  $\square$



**Lemma 4.7.** *<ipseq> Let  $r, s, t$  be the three roots from an offender triple (in any order). The unordered set of inner products  $(w, r), (w, s), (w, t)$  is  $0, 0, \pm 1$ . The unordered set of norms for  $x, y, z$  is one of  $6, 6, 4$  or  $6, 6, 8$ .*

**Proof.** The second statement follows from the first, which we now prove. Let  $r' \in \{r, -r\}$  satisfy  $(w, r') \leq 0$ . Similarly, let  $s' \in \{s, -s\}$  satisfy  $(w, s') \leq 0$  and  $t' \in \{t, -t\}$  satisfy  $(w, t') \leq 0$ . Then  $w + r' + s' + t' \in M \cap N$  and  $w + r' + s' + t'$  has norm  $4 + 2 + 2 + 2 + e$ , where  $e \leq 0$  and  $e$  is even.

We observe that if  $w + r' + s' + t'$  were 0, the pairwise orthogonality of  $r, s, t$  would imply that  $w$  has norm 6, which is not the case. Therefore,  $w + r' + s' + t'$  has even norm at least 8. Consequently,  $e = 0$  or  $e = -2$ . Since  $M \cap N \cong \sqrt{2}\Lambda$ , in which norms are divisible by 4 and nonzero norms are at least 8,  $e = -2$ . Therefore all but one of  $(w, r), (w, s), (w, t)$  is 0 and the remaining one is  $\pm 1$ .  $\square$

**Notation 4.8.** *<super> An offender  $(w, x, y, z)$  is a *super offender* if  $w$  has norm 4 and the norms of  $x, y, z$  in some order are 6, 6, 4.*

**Lemma 4.9.** *<44> We may assume that an offender  $(w, x, y, z)$  satisfies  $(w, w) = 4$ ,  $(w, t) = 1$  and  $(z, z) = 4$ . In other words, if an offender exists, a super offender exists.*

**Proof.** Since  $(w, t) = \pm 1$ ,  $z = t - w$  has norm 4 or 8, respectively. Suppose the latter. Then  $(-w, -x, -y, z + 2w)$  is admissible and its final component  $z + 2w = t + w$  has norm 4. Therefore,  $(-w, -x, -y, z + 2w)$  is a super offender.  $\square$

**Theorem 4.10.** *<6or8> Let  $L := L(M, N)$ , where  $M \cong N$  are isometric to the Leech lattice. Then the minimum norm of  $L$  is 6 if and only if there exists a super offender. Otherwise, the minimum norm is 8.*

**Remark 4.11.** *<conclusion> Given  $M, N$ , (4.10) indicates that checking a (very large) finite number of inner products will settle  $\mu(L(M, N, 3))$ .*

There are finitely many polarizations  $M, N$  of  $\Upsilon$ . Possibly some  $L(M, N, 3)$  have minimum norm 6 and others have minimum norm 8.

Use of isometry groups and other theory might reduce the number of computations significantly.

## 5 Some higher dimensionss

**Lemma 5.1.** *<gen3.5> There exist rank 32 even integral unimodular lattices  $U, M, N$  so that  $\mu(U) = \mu(M) = 4$ ,  $\mu(N) \in \{2, 4\}$  and  $\sqrt{2}M, \sqrt{2}N$  is a polarization of  $U$ .*

**Proof.** We take  $U$  to be  $BW_{2^5}$ . If  $f$  is a fourvolution in  $O(U)$ , then  $M := (f-1)U \cong \sqrt{2}U$ . Therefore, the natural  $\mathbb{F}_2$ -valued quadratic form on  $U/2U$  is split (i.e., has maximal Witt index) and so there exists an even unimodular lattice  $N$  so that  $\sqrt{2}N$  is between  $U$  and  $2U$  and  $\sqrt{2}N/2U$  complements  $M/2U$  in  $U/2U$ . The extremal bound  $\mu(N) \leq 4$  and evenness of  $N$  imply the last statement.  $\square$

We now exhibit a few even unimodular lattices for which the minimum norm is moderately close to the extremal bound  $2(1 + \lfloor \frac{\text{rank}(L)}{24} \rfloor)$ .

**Proposition 5.2.** *<gen4> Let  $U, M, N$  be as in (5.1) and let  $k = 3$ . Then the minimum norm of the rank 96 lattice  $L(M, N, 3)$  is 6 or 8.*

**Proof.** The value of  $\mu$  depends on whether there exists rank 32 even unimodular lattices  $U, M, N$  as in (5.1) so that  $\mu(N) = 4$ .  $\square$

**Theorem 5.3.** *<gen5> There exists an even unimodular lattice  $L(M, N, k)$  of rank  $\ell$  and minimum norm  $\mu$  for the following pairs  $(\ell, \mu)$ :*

- (i)  $(96, 8)$  (the extremal bound is 10);
- (ii)  $(120, 8)$  (the extremal bound is 12).
- (iii)  $(128, 8)$  (the extremal bound is 12)

**Proof.** We use (2.9).

- (i) Take  $k = 4$  and  $U, M, N \cong \Lambda$  (7.3).
- (ii) Take  $k = 5$  and  $U, M, N \cong \Lambda$  (7.3).
- (iii) Take  $k = 4$  where  $U, M, N$  are rank 32 lattices as in (5.1).  $\square$

## 6 Appendix: the index-determinant formula

**Theorem 6.1.** *<indexdet> (“Index-determinant formula”) Let  $L$  be a rational lattice, and  $M$  a sublattice of  $L$  of finite index  $|L : M|$ . Then*

$$\det(L) |L : M|^2 = \det(M).$$

**Proof.** This is a well-known result. Choose a basis  $x_1, \dots, x_n$  for  $L$  and positive integers  $d_1, d_2, \dots, d_n$ , so that  $M$  has a basis  $d_1x_1, d_2x_2, \dots, d_nx_n$ . A Gram matrix for the lattice  $M$  is  $G_M = ((d_ix_i, d_jx_j)) = DG_LD$ , where

$$D = \begin{pmatrix} d_1 & & \\ & d_2 & \\ & & \ddots \end{pmatrix},$$

and  $G_L = ((x_i, x_j))$  is a Gram matrix for  $L$ . Thus  $\det(G_M) = \det(D)^2 \cdot \det(G_L)$ .  $\square$

## 7 Appendix: about fourvolution type sublattices and polarizations of Leech

**Definition 7.1.** *<fourvolution>* A fourvolution  $f$  is a linear transformation whose square is  $-1$ . If  $f$  is orthogonal,  $f - 1$  doubles norms.

**Definition 7.2.** *<fourvolutiontype>* Let  $L$  be an integral lattice. A sublattice  $M$  of  $L$  is of *fourvolution type* if there exists a fourvolution  $f$  so that  $M = L(f - 1)$  (whence  $M \cong \sqrt{2}L$ ). The same terminology applies to scaled copies of  $\Lambda$ .

**Lemma 7.3.** *<leechleechpolar>* If  $U \cong \Lambda$ , there are polarizations of  $\Upsilon$  by sublattices  $M \cong N \cong \Lambda$ .

**Proof.** Here is one proof. We use a fact about  $O(\Lambda)$ , that there are pairs of fourvolutions  $f, g$  so that  $\langle f, g \rangle$  is a double cover of a dihedral group of order  $2k$  for which an element of odd order  $k > 1$  has no eigenvalue 1 on  $\Lambda$ . There exist examples of this for  $k = 3, 5$ , at least (for which  $C_{O(\Lambda)}(\langle f, g \rangle) \cong 2 \cdot G_2(4), 2 \cdot HJ$ , respectively) [6]. We take  $M := \Lambda(f - 1)$  and  $N := \Lambda(g - 1)$ . Since  $2\Lambda = \Lambda(f - 1)^2 = \Lambda(g - 1)^2$ ,  $M \cap N \geq 2\Lambda$ . We argue that the pair  $M, N$  gives a polarization. Since  $(M \cap N)/2\Lambda$  consists of vectors fixed by  $\langle f, g \rangle$ , it is 0. By determinant considerations,  $M + N = \Upsilon$ .  $\square$

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